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ENUMERATION OF CONES OVER CUBIC SCROLLS

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ABSTRACT

We enumerate the cones over cubic scrolls satisfying appropriate incidence conditions to lines.

1. Introduction

The enumeration of rational curves has attracted much attention, from the ancient's circles of Apollonius, to the surprising connection with string theory in the last two decades (cf. [5]). More recently, the enumeration of certain rational surfaces was the main subject of I. Coskun's thesis [1]. He relies on degeneration techniques; these work satisfactorily, provided one imposes the condition of incidence to sufficiently many points. It seems to become quite

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involved and ineffective otherwise, e.g., for the case of incidence to the maximal number of lines, say for the family of cubic scrolls in $\mathbb{P}^4$. The latter in turn can be handled by more classical tools, e.g., [8].

We consider here the question of enumerating families of codimension two cones over a cubic scroll S_i of dimension $i = 1, 2, 3$. A nondegenerate S_i is defined in $\mathbb{P}^{2+i}$ by the 2×2 minors of a 2×3 matrix of general linear homogeneous polynomials. For the sake of concreteness we describe the construction for the 18-dimensional family of cones in $\mathbb{P}^5$ over a varying cubic surface scroll S_2 (*csc* for short). These are precisely the irreducible, singular cubic threefolds that span $\mathbb{P}^5$, cf. [10]. This family includes as members the joins of a twisted cubic and a line, the enumeration of which is also done. The effective calculation of numbers is performed using a MAPLE script [9] that produces the desired numbers for $i = 1, 2, 3$ and arbitrary vertex dimension; a sample is reproduced in a table at the end of this note. Obvious variations will work for families of cones over any family of varieties for which a manageable parameter space is available.

Here is a brief outline. First we review, for the reader's ease, the well-known relationship between a cone with a given vertex $V \simeq \mathbb{P}^v \subset \mathbb{P}^n$ and its abstract base in the grassmannian $\mathbb{G}\mathrm{r}[v+1, n]$ of subspaces of $\mathbb{P}^n$ of dimension $v+1$. The base actually lies in the projectivization of the normal space to V, embedded in that grassmannian as a "star" of all $\mathbb{P}^{v+1} \subset \mathbb{P}^n$ containing V. The condition of incidence of a codimension two cone to a line in $\mathbb{P}^n$ is rephrased in terms of the condition of incidence of the abstract base of the cone to a line *within* the grassmannian. Next, borrowing from [7], [8] we summarize the construction of a suitable parameter space for the family of cubic surface scrolls in $\mathbb{P}^4$. This is aimed at the precise geometric characterization of blowup centers and natural vector bundles needed to feed into Bott's residue formula. We learned the version of Bott's formula used here from P. Meurer [4] and D. Edidin and W. Graham [2]. In §4 we describe the cycles in terms of Chern classes of natural bundles. These will be evaluated in equivariant cohomology at the fixed points as described in the last section.

2. Cones

2.1 NOTATION. We denote by $[m]$ a projective subspace of dimension m in $\mathbb{P}^n$.

Our basic setup builds a cone, say over a variety B of dimension b which spans $\langle B\rangle = [b+2] \subset \mathbb{P}^n$, with vertex $V = [v] \subset \mathbb{P}^n \setminus \langle B\rangle$. We assume, for simplicity of exposition,

$$n = v + b + 3. \tag{1}$$

The example to keep in mind is that of a cubic scroll surface in $\mathbb{P}^4$, so that we have $b = 2, v = 0, n = 5$.

2.1.1 Vertices. We consider the grassmannian of vertices, $\mathbb{G}\mathrm{r}[v, n]$, with respective tautological sequence written as

$$\mathcal{V} \rightarrowtail \mathcal{O}^{\oplus(n+1)} \twoheadrightarrow \mathcal{T} \tag{2}$$

where rank $\mathcal{V} = v + 1$ and rank $\mathcal{T} = n - v = b + 3$.

2.2.2 Generators. The set of all $[v + 1]$ in $\mathbb{P}^n$ through a given $V = [v]$ is, of course, in bijection with any subspace $[b + 2] \subset \mathbb{P}^n$ disjoint from V. This set is identified to a specific projective space $\mathbb{P}_V^{b+2}$, naturally embedded in the grassmannian $\mathbb{G}\mathrm{r}[v + 1, n]$ of all $[v + 1] \subset \mathbb{P}^n$.

We have the tautological exact sequence of vector bundles over $\mathbb{G}\mathrm{r}[v + 1, n]$,

$$\mathcal{S} \rightarrowtail \mathcal{O}^{\oplus(n+1)} \twoheadrightarrow \mathcal{R} \tag{3}$$

where rank $\mathcal{S} = v + 2$ and rank $\mathcal{R} = n - v - 1 = b + 2$.

We will time and again reason over the following diagram of incidence, with natural maps

$$\begin{tikzcd}[column sep=small] & \mathbb{G}\mathrm{r}[v,n]\times\mathbb{G}\mathrm{r}[v+1,n] \arrow[d, phantom, "\cup"] & \\ & \mathbb{P}(\mathcal{T}) = \mathbb{P}(\mathcal{S}^\vee) \supset \lambda^{-1}\mathbb{P}_V^{b+2} \arrow[dl, "\beta"'] \arrow[dr, "\lambda"] & \\ \mathbb{G}\mathrm{r}[v,n] & & \mathbb{G}\mathrm{r}[v+1,n] \supset \mathbb{P}_V^{b+2} \supset B. \end{tikzcd} \tag{4}$$

Here $\lambda : \mathbb{P}(\mathcal{S}^\vee) \to \mathbb{G}\mathrm{r}[v + 1, n]$ is the dual of the universal $\mathbb{P}^{v+1}$-bundle over $\mathbb{G}\mathrm{r}[v + 1, n]$. The total space $\mathbb{P}(\mathcal{S}^\vee)$ is naturally isomorphic to the total space of the projective bundle

$$\beta : \mathbb{P}(\mathcal{T}) \to \mathbb{G}\mathrm{r}[v, n].$$

The isomorphism $\mathbb{P}(\mathcal{S}^\vee) \simeq \mathbb{P}(\mathcal{T})$ is encoded by the following diagram of vector bundles

$$\begin{tikzcd} \mathcal{V} \arrow[r, tail] \arrow[d, equal] & \mathcal{S} \arrow[r, two heads] \arrow[d, tail] & \mathcal{O}_{\mathcal{T}}(-1) \arrow[d, tail] \\ \mathcal{V} \arrow[r, tail] & \mathcal{O}^{\oplus(n+1)} \arrow[r, two heads] \arrow[d, two heads] & \mathcal{T} \arrow[d, two heads] \\ & \mathcal{R} \arrow[r, equal] & \mathcal{R} \end{tikzcd}$$

where the middle vertical sequence is (3) whereas the middle horizontal one comes from (2). The fiber $\mathbb{P}_V^{b+2} = \beta^{-1}V \subset \mathbb{G}\mathrm{r}[v+1,n]$ is the set of all $[v+1] \subset \mathbb{P}^n$ through a given $V = [v]$.

2.1.3 Bases. Let $\mu : \mathbb{P}(\mathcal{S}) \to \mathbb{G}\mathrm{r}[v+1,n]$ be the universal $[v+1]$-bundle. Let $\psi : \mathbb{P}(\mathcal{S}) \to \mathbb{P}^n$ be the natural projection map. One checks that the restriction, ψ_V of ψ to $\mu^{-1}\mathbb{P}_V^{b+2}$ is the blowup of $\mathbb{P}^n$ along V. Given a subspace $[b+2] \subset \mathbb{P}^n$ disjoint from V, we have that ${\psi_V}^{-1}[b+2]$ maps isomorphically to $\mathbb{P}_V^{b+2}$ via μ. If $B \subset \mathbb{P}_V^{b+2}$ is a subvariety of dimension b, the image of the $\mathbb{P}^{v+1}$-bundle $\mu^{-1}B$ via ψ,

$$S_{B,V} := \psi(\mu^{-1}B) \subset \mathbb{P}^n$$

is the corresponding codimension 2 cone with abstract base B and vertex V. This establishes an isomorphism between a base $[b+2] \cap S_{B,V} \subset \mathbb{P}^n$ of the cone and the (abstract) base $B \subset \mathbb{P}_V^{b+2} \subset \mathbb{G}\mathrm{r}[v+1,n]$.

2.1.4 Incidence. Fix a line $\ell_0 = [1] \subset \mathbb{P}^n$. The condition that the cone $S_{B,V}$ meet ℓ_0 is equivalent to the condition that some $[v+1]$ in $S_{B,V}$ through V meet ℓ_0. Thus, we may interpret incidence of a cone to a line in $\mathbb{P}^n$ as a condition *within* $\mathbb{G}\mathrm{r}[v+1,n]$: namely, require that the base $B \subset \mathbb{P}_V^{b+2} \subset \mathbb{G}\mathrm{r}[v+1,n]$ meet the Schubert cycle $\ell_0^\star$ of all $[v+1]$ incident to our fixed ℓ_0. Now, as long as the line ℓ_0 does not meet the vertex V, the intersection $\ell_0^\star \cap \mathbb{P}_V^{b+2}$ is easily seen to be a line $\ell_{0V}^\star$ in $\mathbb{P}_V^{b+2}$:

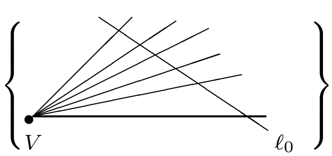

Figure 1. The pencil of lines $\ell_{0V}^\star$.

This is the pencil of all $[v+1]$ in $\mathbb{P}^n$ made up by the joins of the vertex V with the points on ℓ_0. See Fig. 1.

In short, we have got a family, $\mathbb{P}(\mathcal{T}) \to \mathbb{G}\mathrm{r}[v,n]$, of $[b+2]$'s moving inside the grassmannian $\mathbb{G}\mathrm{r}[v+1,n]$. Thus, if we are given a family of "bases" $B_{t,V} \subset \mathbb{P}_V^{b+2}$, we may study the cycle defined (in a suitable parameter space $\mathfrak{T} \ni t$) by the condition that $B_{t,V}$ meet the line $\ell_{0V}^\star$.

3. A parameter space of cubic scrolls

We assume henceforth, for the sake of concreteness, that the bases of the cones are cubic surface scrolls. Thus, we have $b = 2$ in (1). Recall that any two quadrics containing a cubic scroll cut a residual codimension-2 subspace. The main idea of the construction of the parameter space in [8] summarized below is to reverse the process: look at pencils of quadrics through a codimension-2 subspace. A Hilbert scheme construction is also available, though less elementary, cf. [6], [3].

Let $\mathcal{E}\to X$ be a vector bundle of rank 5 over a smooth variety X.

We will eventualy take $\mathcal{E}$ to be the tautological rank 5 quotient bundle $\mathcal{T}$ over $X = \mathbb{G}\mathrm{r}[v, n], n = v + 5$.

Put $\mathcal{F}=\mathcal{E}^\vee$. Form the Grassmann bundle

$$\mathbb{G}\mathrm{r}(2, \mathcal{F})\longrightarrow X$$

of rank-2 vector subspaces of the fibers of $\mathcal{F}\to X$. This is the same as the bundle $\mathbb{G}\mathrm{r}(3, \mathcal{E})$ of codimension-2 subspaces in the fibers of $\mathbb{P}(\mathcal{E})\to X$, via the natural duality identification,

$$\mathbb{G}\mathrm{r}(2, \mathcal{F}) \ni \ell \leftrightarrow \ell^\perp \in \mathbb{G}\mathrm{r}(3, \mathcal{E}).$$

We write

$$\mathcal{U} \rightarrowtail \mathcal{F} \twoheadrightarrow \mathcal{Q} \tag{5}$$

for the tautological sequence over $\mathbb{G}\mathrm{r}(2, \mathcal{F})$, where rank $\mathcal{U}$=2, rank $\mathcal{Q}$=3. The fiber of $\mathcal{U}$ over a point $\ell \in \mathbb{G}\mathrm{r}(2, \mathcal{F})$ which lies over some $x \in X$ is the space of linear forms on $\mathbb{P}^4 = \mathbb{P}(\mathcal{E}_x)$ vanishing on the codimension 2 subspace $\pi = \ell^\perp \subset \mathbb{P}(\mathcal{E}_x)$ dual to ℓ.

We now consider the bundle of forms of degree m vanishing along a varying codimension 2 subspace π,

$$\mathcal{F}_m^\perp = \ker (S_m\mathcal{F} \twoheadrightarrow S_m\mathcal{Q}) \tag{6}$$

where S_m denotes symmetric power. Note that

$$\operatorname{rank} \mathcal{F}_m^\perp = \binom{m+4}{m} - \binom{m+2}{m}. \tag{7}$$

Next we take the Grassmann bundle

$$\mathbb{X} := \mathbb{G}\mathrm{r}(2, \mathcal{F}_2^\perp)\longrightarrow \mathbb{G}\mathrm{r}(2, \mathcal{F}) \tag{8}$$

with corresponding tautological sequence

$$\mathcal{A} \rightarrowtail \mathcal{F}_2^{\perp} \twoheadrightarrow \mathcal{Q}_{\mathcal{A}}. \tag{9}$$

A point $\mathrm{x} \in \mathbb{X}$ lying over $\ell \in \mathbb{G}\mathrm{r}(2, \mathcal{F})$ is a pencil of quadrics containing the distinguished codimension 2 subspace $\pi = \ell^{\perp}$ in the projective space $\mathbb{P}(\mathcal{E}_x) \simeq \mathbb{P}^4$, where x denotes the image of x in X. We write $\mathrm{x} = (x, \ell, \langle q_1, q_2 \rangle)$.

The relative dimension of $\mathbb{X}$ over X is

$$20 = \overbrace{6}^{\text{rel.dim } \mathbb{G}\mathrm{r}(2,\mathcal{F})} + \overbrace{2 \cdot (9-2)}^{\text{rel.dim } \mathbb{G}\mathrm{r}(2,\mathcal{F}_2^{\perp})} .$$

This is 2 in excess to the well-known dimension of the Hilbert scheme of cubic scrolls in $\mathbb{P}^4$. The excess is due to the assignment the ∞^2-many 2-planes residual to a complete intersection of 2 quadrics through a *csc*. In practice, we should eventually restrict $\mathbb{X}$, as well as the modifications performed thereon, over some suitable subscheme of codimension 2 in $\mathbb{G}\mathrm{r}(2, \mathcal{F})$, in such a way that a general point of $\mathbb{X}$ correspond to a unique *csc*. This is taken care of in §4.

Continuing, we recall from [7] (see also [8]) the construction of a parameter space for the family of *csc*s in the fibers of a $\mathbb{P}^4$-fibration, $\mathbb{P}(\mathcal{E}) \to X$. It is obtained by a sequence of three explicit blowups,

$$\mathbb{X}''' \longrightarrow \mathbb{X}'' \longrightarrow \mathbb{X}' \longrightarrow \mathbb{X}, \tag{10}$$

with $\mathbb{X}$ as in (8) above. The first blowup center is the subscheme $\mathbb{Y}$ of $\mathbb{X}$ consisting of all $\mathrm{x} = (x, \ell, \langle q_1, q_2 \rangle)$ parametrizing the pencils of quadrics of the form $\langle h \cdot h_1, h \cdot h_2 \rangle$ with a common component a hyperplane $h \supset \pi = \ell^{\perp}$:

$$\mathbb{Y} = \{(x, \ell, \langle h \cdot h_1, h \cdot h_2 \rangle) \in \mathbb{X} |\ h \supset \ell^{\perp}\}. \tag{11}$$

The residual pencil of hyperplanes, $\langle h_1, h_2 \rangle$, defines another codimension 2 subspace, $\ell'^{\perp}$, as depicted below. Beware, this is supposed to be a configuration in $\mathbb{P}^4$:

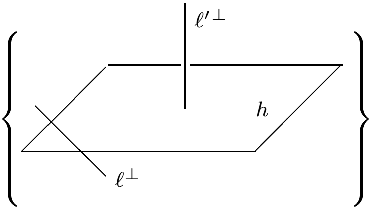

Figure 2. The first blowup center $\mathbb{Y}$.

Observe that $\mathbb{Y}$ is the isomorphic image in $\mathbb{X}$ of the fiber product,

$$\mathbb{Y} \simeq \mathbb{P}(\mathcal{U}) \times_X \mathbb{G}\mathrm{r}(2, \mathcal{F}). \tag{12}$$

The first factor, $\mathbb{P}(\mathcal{U})$, parametrizes the partial flags $\pi = \ell^\perp \subset h$, whereas the second one, $\mathbb{G}\mathrm{r}(2, \mathcal{F})$, accounts for the codimension 2 subspace $\ell'^\perp$. The relative dimension is

$$\dim_X \mathbb{Y} = \overbrace{4+3}^{h \supset \ell^\perp} + \overbrace{\ 6\ }^{\ell'} = 13. \tag{13}$$

The subscheme $\mathbb{Y}$ turns out to be the indeterminacy locus of a natural rational map $\mathbb{X} \dashrightarrow \mathbb{G}\mathrm{r}(3, \mathcal{F}_2)$ which assigns to a general pencil of quadrics $\langle q_1, q_2 \rangle$ through a distinguished codimension 2 subspace $\ell^\perp$, the net $\langle q_1, q_2, q_3 \rangle$ of quadrics cutting the residual *csc*.

We register two other interesting subschemes of $\mathbb{X}$. Let $\mathcal{B}$ be the rank-4 vector bundle over $\mathbb{P}(\mathcal{U})$ defined as the cokernel of the composition

$$\mathcal{O}_{\mathcal{U}}(-1) \rightarrowtail \mathcal{U} \rightarrowtail \mathcal{F}.$$

The fiber of $\mathcal{B}$ over a point $(x, \ell^\perp \subset h) \in \mathbb{P}(\mathcal{U})$ can be thought of as the space of linear forms on the hyperplane h.

We set

$$\begin{aligned} &\mathbb{Y}_1 \simeq \mathbb{P}(\mathcal{F}) \times_X \mathbb{G}\mathrm{r}(2, \mathcal{F}); \\ &\mathbb{Y}_2 \simeq \mathbb{P}(\mathcal{B}). \end{aligned} \tag{14}$$

Each point $(x, h, \ell) \in \mathbb{P}(\mathcal{F}) \times_X \mathbb{G}\mathrm{r}(2, \mathcal{F})$ is mapped to $(x, \ell, h \cdot \ell) \in \mathbb{X}$, with the obvious abuse $\ell = \mathcal{U}_\ell$, the 2-dimensional space of linear forms that cut $\ell^\perp \subset \mathbb{P}^4 = \mathbb{P}(\mathcal{E}_x)$. The intersection $\mathbb{Y} \cap \mathbb{Y}_1$ is isomorphic to the incidence subscheme of $\mathbb{Y}_1$ where $h \supset \ell^\perp$. It is also the locus in $\mathbb{Y}$ where the distinguished codimension 2 subspace $\ell^\perp$ and the residual codimension 2 subspace $\ell'^\perp$ coincide (see Fig. 2). Each (closed) point of $\mathbb{Y}_2$ lying over a flag $(\ell^\perp \subset h) \in \mathbb{P}(\mathcal{U})$ corresponds to a rank-2 subspace of the corresponding fiber of $\mathcal{F}$, that is, a codimension 2 subspace $\ell'^\perp \subset h$. Thus, we have

$$\mathbb{Y}_2 = \{(\ell, h, \ell') \in \mathbb{P}(\mathcal{U}) \times_X \mathbb{G}\mathrm{r}(2, \mathcal{F}) |\ \ell^\perp \subset h \supset \ell'^\perp\} \subset \mathbb{Y}. \tag{15}$$

The embedding of $\mathbb{Y}_2$ in $\mathbb{X}$ maps (ℓ, h, ℓ') to the pencil $h \cdot \ell'$ (with distinguished codimension 2 subspace $\ell^\perp \subset h$).

Blowing up $\mathbb{X}$ along $\mathbb{Y}$ yields a smooth subvariety $\mathbb{X}' \subset \mathbb{X} \times_X \mathbb{G}\mathrm{r}(3, \mathcal{F}_2)$. A general point in a fiber of the exceptional divisor over $\langle h \cdot h_1, h \cdot h_2 \rangle$ corresponds to the configuration given by the codimension 2 subspace $\langle h_1, h_2\rangle$ union a quadric in the hyperplane h passing through the (codimension 3) intersection $\langle h, h_1, h_2\rangle$.

Each point of $\mathbb{X}'$ corresponds to a flag $A \subset B \subset \mathcal{F}_2$ such that B is a determinantal net and A is a pencil of quadrics through a distinguished codimension 2 subspace. We have the vector bundle maps defined fiberwise by multiplication, $A \otimes \mathcal{F} \xrightarrow{\alpha} \mathcal{F}_3$ and $B \otimes \mathcal{F} \xrightarrow{\beta} \mathcal{F}_3$; their generic ranks are 10 and 13 respectively. Indeed, the number of conditions that a complete intersection of two quadrics, resp. a *csc*, imposes on cubics is equal to 35–10, resp. 35–13. It is shown in [8] that the Fitting subscheme $\mathbb{Y}'$ of $\mathbb{X}'$ where the rank of β drops to 12 consists of 2 smooth components, denoted $\mathbb{Y}'_1, \mathbb{Y}'_2$.

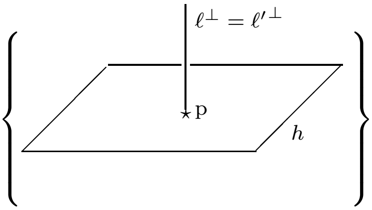

Figure 3. The 2nd blowup center, $\mathbb{Y}'_1$

The first component is isomorphic to the incidence variety

$$\mathbb{Y}'_1 \cong \{(\mathrm{p}, \ell, h) \in \mathbb{G}\mathrm{r}(2, \mathcal{E}) \times_X \mathbb{G}\mathrm{r}(3, \mathcal{E}) \times_X \mathbb{P}(\mathcal{F}) \,|\, \mathrm{p} \subseteq h \cap \ell^\perp \subset \mathbb{P}^4\}. \tag{16}$$

$\mathbb{Y}'_1$ turns out to be the locus where the rank of α drops to 9. It is the strict transform of $\mathbb{Y}_1$, (14). The image of α over a point $(\mathrm{p}, \ell, h) \in \mathbb{Y}'_1$ is the space of cubics $h \cdot \mathcal{F}_2^\ell$ of multiples of quadrics vanishing on $\ell^\perp$. If the codimension 3 subspace p is cut out by the net of hyperplanes $\langle h_0, h_1, h_2\rangle$ with $\ell = \langle h_1, h_2\rangle$, the embedding $\mathbb{Y}'_1 \subset \mathbb{X}'$ maps (p, ℓ, h) to the flag

$$\langle h \cdot h_1, h \cdot h_2\rangle \subset \langle hh_0, h \cdot h_1, h \cdot h_2\rangle.$$

Notice that the scheme defined by the latter net of quadrics is a hyperplane together with an embedded component supported at a codimension 3 subspace in the intersection with the distinguished codimension 2 subspace. We have $\dim_X \mathbb{Y}'_1 = 4 + 6$.

The second component, $\mathbb{Y}'_2$, is a $\mathbb{P}^2$-subbundle of the restriction of the exceptional divisor over $\mathbb{Y}_2$, (15). It can be described as

$$\mathbb{Y}'_2 \cong \{(\mathrm{p}, \ell, \ell', h) \in \mathbb{G}\mathrm{r}(2, \mathcal{E}) \times_X \mathbb{G}\mathrm{r}(2, \mathcal{F}) \times_X \mathbb{P}(\mathcal{U}) \,|\, \mathrm{p} \subset \ell'^\perp \subset h \supset \ell^\perp\}.$$

The embedding

$$\mathbb{Y}_2' \subset \mathbb{X} \times_X \mathbb{G}\mathrm{r}(3, \mathcal{F}_2) \tag{17}$$

assigns to $(\mathrm{p}, \ell, \ell', h)$ the net of quadrics $\langle h^2, h \cdot h_1, h \cdot h_2 \rangle$ so that

$$\mathrm{p} \in \mathbb{G}\mathrm{r}(2, \mathcal{E}) = \mathbb{G}\mathrm{r}(3, \mathcal{F})$$

is defined by the net of hyperplanes $\langle h, h_1, h_2 \rangle$. We have $\dim_X \mathbb{Y}_2' = 12$.

Let $\mathbb{X}''$ be the blow up of $\mathbb{Y}_1' \subset \mathbb{X}'$. It embeds in $\mathbb{X}' \times_{\mathbb{G}\mathrm{r}(2,\mathcal{F})} \mathbb{G}\mathrm{r}(10, \mathcal{F}_3^{\perp})$ as the closure of the graph of the rational map defined by $\langle q_1, q_2 \rangle \mapsto \langle q_1, q_2 \rangle \cdot \mathcal{F}$.

Consider the strict transform of $\mathbb{Y}_2'$,

$$\mathbb{Y}_2'' \cong \{(\mathrm{p}, \mathrm{o}, \ell, h, \ell') \in \mathbb{G}\mathrm{r}(2, \mathcal{E})^{\times 2} \times_X \mathbb{G}\mathrm{r}(2, \mathcal{F}) \times_X \mathbb{P}(\mathcal{U}) \mid \mathrm{p}, \mathrm{o} \subset \ell'^{\perp} \subset h \supset \ell^{\perp} \supset \mathrm{o}\}. \tag{18}$$

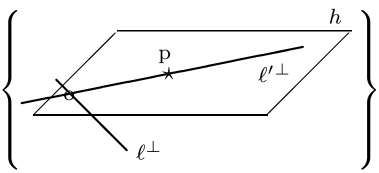

Figure 4. The last blowup center, $\mathbb{Y}_2''$

For $\mathrm{y}'' = (\mathrm{p}, \mathrm{o}, \ell, h, \ell') \in \mathbb{Y}_2''$, let $\mathcal{F}_2^{(\mathrm{o}+\mathrm{p}, \ell')}$ be the 10-dimensional space of quadrics containing the line-pair $\mathrm{o}, \mathrm{p} \subset \ell'^{\perp}$. Then the embedding of $\mathbb{Y}_2''$ in $\mathbb{X}' \times_{\mathbb{G}\mathrm{r}(2,\mathcal{F})} \mathbb{G}\mathrm{r}(10, \mathcal{F}_3^{\perp})$ assigns to the point y'' the 10-dimensional space of cubics $h \cdot \mathcal{F}_2^{(\mathrm{o}+\mathrm{p}, \ell')}$. Let $\mathcal{C} \subset \mathcal{F}_3$ denote the image sheaf of the map β defined over $\mathbb{X}'$ by multiplication of a net of quadrics by the sheaf $\mathcal{F} \otimes \mathcal{O}_{\mathbb{X}'}$ of linear forms. Let $\widetilde{\mathcal{C}} \subset \mathcal{F}_3$ denote the **saturation** of the image of $\mathcal{C} \otimes \mathcal{O}_{\mathbb{X}''} \to \mathcal{F}_3$, i.e., the inverse image of the torsion subsheaf of $\mathcal{F}_3/\mathcal{C}$. The inclusion map $\widetilde{\mathcal{C}} \subset \mathcal{F}_3$ is of generic rank 13. The rank drops to 12 precisely along $\mathbb{Y}_2''$. The image of $\widetilde{\mathcal{C}} \subset \mathcal{F}_3$ over a point $\mathrm{y}'' \in \mathbb{Y}_2''$ as above is the space of cubics $h \cdot \mathcal{F}_2^{\mathrm{p}}$. (The codimension-3 subspace $\mathrm{p} \subset \mathbb{P}^4$ imposes 3 conditions on the 15 dimensional vector space of quadrics in $\mathbb{P}^4$.)

Blowing up $\mathbb{Y}_2'' \subset \mathbb{X}''$ produces the smooth compactification

$$\mathbb{X}''' \subset \mathbb{X} \times_X \mathbb{G}\mathrm{r}(3, \mathcal{F}_2) \times_X \mathbb{G}\mathrm{r}(10, \mathcal{F}_3) \times_X \mathbb{G}\mathrm{r}(13, \mathcal{F}_3) \tag{19}$$

studied in [7] and [8].

The exceptional divisors are described in the sequel. This is required for the determination of fixed points that enter Bott's formula (29).

The exceptional fiber of $\mathbb{E}' \to \mathbb{Y}$ over a point $(\ell, h, \ell') \in \mathbb{Y}$ is the projectivization of the space of quadrics vanishing on the plane $\ell'^{\perp}$, modulo the space $h \cdot \ell'$. In symbols,

$$\mathbb{E}'_{(\ell,h,\ell')} = \mathbb{P}\left(\mathcal{F}_2^{\ell'}/(h \cdot \ell')\right). \tag{20}$$

A general fiber corresponds to the configuration formed by $\ell'^{\perp} \simeq \mathbb{P}^2 \subset \mathbb{P}^4$ union a quadric surface in the hyperplane h containing the line $h \cap \ell'^{\perp}$. The fiber dimension is, of course, equal to $6(= 15 - 6 - 2 - 1)$, cf. (7), as expected for the projectivization of the normal space to $\mathbb{Y} \subset \mathbb{X}$, which has rank $7(= 20 - 13)$, see (13).

The fiber of the second exceptional divisor, $\mathbb{E}'' \to \mathbb{Y}'_1$ over (p, ℓ, h) (see (16)) is the projectivization of the quotient vector space

$$\mathcal{F}_3^{\ell+\mathrm{p}^2}/(h \cdot \mathcal{F}_2^{\ell}). \tag{21}$$

That is the space of cubics containing the plane $\ell^{\perp}$, singular along the line $\mathrm{p} \subseteq \ell^{\perp} \cap h$, modulo h times the space the quadrics through $\ell^{\perp}$. Fiber dimension count: $19 - 9 - 1 = 9$; codimension count: $20 - \dim_X \mathbb{Y}'_1 = 10$.

To describe the fiber of the last exceptional divisor $\mathbb{E}''' \to \mathbb{Y}''_2$ over $(\mathrm{p}, \mathrm{o}, \ell, h, \ell') \in \mathbb{Y}''_2$, we first note that $\mathbb{Y}''_2$ carries a flat family of closed subschemes of $\mathbb{P}^4$ with general member defined by the intersection of the ideal sheaf of the line o and the square of the ideal sheaf of the line p. Any such subscheme has Hilbert polynomial equal to $5t$ and imposes independent conditions on cubics. Since the lines o and p are constrained to vary inside the plane $\ell'^{\perp}$, we still get at the limit, when o=p, a well-defined subscheme with the same Hilbert polynomial, supported at the line. For instance, say

$$h = z_0,\ \ell = \ell' = \langle z_0, z_1 \rangle,\ \mathrm{o} = \mathrm{p} = \langle z_0, z_1, z_2 \rangle.$$

The scheme structure is defined by the ideal

$$\langle z_2^3, z_2 z_1, z_2 z_0 \rangle + \langle z_0, z_1 \rangle^2.$$

If we keep h, ℓ', p as above but set $\mathrm{o} = \langle z_0, z_1, z_3 \rangle \neq \mathrm{p}$, we find the cubic generator is replaced by $z_2^2 z_3$. Let $\mathcal{F}_3^{\ell',\mathrm{o}+\mathrm{p}^2}$ denote the corresponding 20–dimensional space of cubics; that is the space of cubics containing the line o, singular along the line $\mathrm{p} \subset \ell'^{\perp}$. The space $\mathcal{F}_3^{\ell',\mathrm{o}+\mathrm{p}^2}$ contains h times the space, $\mathcal{F}_2^{\mathrm{p}}$, of quadrics through the line p. The exceptional fiber is the projectivization of the quotient vector space

$$\mathcal{F}_3^{\ell',\mathrm{o}+\mathrm{p}^2}/(h \cdot \mathcal{F}_2^{\mathrm{p}}). \tag{22}$$

Fiber dimension count: $19-(15-3)=7$; codimension count: $20-\dim_X \mathbb{Y}_2''=8$.

4. Cycles

First, let us see how to cut the dimension of the parameter space $\mathbb{X}'''$ to the correct size, in such a way that the induced map to the appropriate relative Hilbert scheme component be birational.

For instance, if we were able to get a section of the $\mathbb{P}^4$-bundle $\mathbb{P}(\mathcal{E})\to X$, we could ask the distinguished codimension 2 subspaces on each fiber of $\mathbb{X}\to\mathbb{G}\mathrm{r}(2,\mathcal{F})$ to go through the point marked by that section. More precisely, suppose we have a section

$$x_0'\colon X\to\mathbb{P}(\mathcal{E})$$

defined by a line subbundle $\text{Ł}\rightarrowtail\mathcal{E}$ over X. We may form the codimension 2 relative Schubert cycle

$$\Sigma^2_{x_0}\subset\mathbb{G}\mathrm{r}(2,\mathcal{F})$$

given by the zeros of the composition $\text{Ł}\rightarrowtail\mathcal{E}=\mathcal{F}^\vee\to\mathcal{U}^\vee$. We then have the formula

$$[\Sigma^2_{x_0}]=c_2(\text{Ł}^\vee\otimes\mathcal{U}^\vee)\cap\mathbb{G}\mathrm{r}(2,\mathcal{F}). \tag{23}$$

The example to keep in mind, as we have already mentionned, is as follows. We take $\mathcal{E}=\mathcal{T}\to\mathbb{G}\mathrm{r}[v,n]$ as in (2). We choose the following recipe to pick a point in $\mathbb{P}(\mathcal{T})$ over a general point $V\in\mathbb{G}\mathrm{r}[v,n]$. First note that this is tantamount to choosing a $[v+1]$ in $\mathbb{P}^n$ passing through the given $V=[v]$, see (4). Start by fixing some point $p_0\in\mathbb{P}^n$; then for each $V\in\mathbb{G}\mathrm{r}[v,n], V\not\ni p_0$, the join $\langle p_0,V\rangle$ yields the desired rational section

$$p_0'\colon \mathbb{G}\mathrm{r}[v,n]\dashrightarrow\mathbb{P}(\mathcal{T}).$$

In the diagram with the tautological sequence below, the vertical section defines the given point $p_0\in\mathbb{P}^n$,

$$\begin{tikzcd} & \mathcal{O} \arrow[d, tail, "p_0"'] \arrow[dr] & \\ \mathcal{V} \arrow[r, tail] & \mathcal{O}^{\oplus(n+1)} \arrow[r, two heads] & \mathcal{T}. \end{tikzcd}$$

The slanted arrow $\mathcal{O}\to\mathcal{T}$ is split injective off p_0. Thus, letting

$$p_0^\star=\{V|\ V\ni p_0\},$$

yields the desired section, $\mathcal{O} \rightarrowtail \mathcal{E} = \mathcal{T}$ over $\mathbb{G}\mathrm{r}[v,n] \setminus p_0^\star$. Note that the codimension-2 cycle $\Sigma^2_{p_0} \subset \mathbb{G}\mathrm{r}(2,\mathcal{F})$ just defined represents $c_2(\mathcal{U})$ in the Chow group of the open subset of $\mathbb{G}\mathrm{r}(2,\mathcal{F})$ lying over the domain of regularity of the rational section p_0'. In the example at hand, the rational section p_0' is regular over $\mathbb{G}\mathrm{r}[v,n] \setminus p_0^\star$. By excision, since codim $p_0^\star = 5$, it follows that

$$[\Sigma^2_{p_0}] = c_2(\mathcal{U}^\vee) \cap [\mathbb{G}\mathrm{r}(2,\mathcal{F})] \tag{24}$$

in the Chow group of $\mathbb{G}\mathrm{r}(2,\mathcal{F})$, not just over the open set $\mathbb{G}\mathrm{r}[v,n] \setminus p_0^\star$.

We now describe the codimension one cycle in $\mathbb{G}\mathrm{r}(2,\mathcal{F})$ supported on the closure of the set of codimension 2 subspaces $\ell^\perp \subset \mathbb{P}^4_x$ lying in the fiber over a point $x \in X$ and meeting a suitable varying line $\ell_0'(x) \subset \mathbb{P}^4_x$. Suppose we pick a rational section

$$\ell_0' \colon X \dashrightarrow \mathbb{G}\mathrm{r}(2,\mathcal{E}) = \mathbb{G}\mathrm{r}(3,\mathcal{F}) \tag{25}$$

regular off a codimension-2 subset of X (granted since we take X smooth, hence normal). Say it is defined by an injection of a rank-3 subbundle $\mathcal{U}_0 \rightarrowtail \mathcal{F}$, which we assume to be locally split in codimension-2. Then the sought for codimension-1 cycle class is given by

$$[\Sigma^1_{\ell_0}] = c_1(\textstyle\bigwedge^3 \mathcal{U}_0^\vee \otimes \bigwedge^3 \mathcal{Q}) \cap [\mathbb{G}\mathrm{r}(2,\mathcal{F})]. \tag{26}$$

Indeed, the incidence condition translates into the requirement that the homomorphism $\mathcal{U}_0 \to \mathcal{Q}$ in the diagram below be of rank < 3.

$$\begin{tikzcd} & \mathcal{U}_0 \arrow[d, tail, "p_0"'] \arrow[dr] & \\ \mathcal{U} \arrow[r, tail] & \mathcal{F} \arrow[r, two heads] & \mathcal{Q}. \end{tikzcd}$$

Referring to our pet example, let us fix a line $\ell_0 \subset \mathbb{P}^n$. Let $\ell_0^\star \subset \mathbb{G}\mathrm{r}[v,n]$ be the set of all V incident to ℓ_0. For each $V \notin \ell_0^\star$, the join $\langle V, \ell_0 \rangle$ is a $[n-3]$ in $\mathbb{P}^n$; the set of all $[v+1]$ lying in $\langle V, \ell_0 \rangle$ and passing through V (see Fig. 1) defines a line $\ell_0'(V) \subset \mathbb{P}^{b+2}_V$. Now the support of $\Sigma^1_{\ell_0}$ is the closure of the set of codimension 2 subspaces $\ell^\perp \subset \mathbb{P}^{b+2}_V$ such that $\ell^\perp$ meets $\ell_0'(V)$. Presently, the rational section $\ell_0' \colon \mathbb{G}\mathrm{r}[v,n] \dashrightarrow \mathbb{G}\mathrm{r}(3,\mathcal{F})$ arises from the diagram

$$\begin{tikzcd} \mathcal{U}_0 \arrow[r, tail] \arrow[d, tail] & \mathcal{F} \arrow[r, two heads] \arrow[d, tail] & \mathcal{O}^{\oplus 2} \arrow[d, equal] \\ \mathcal{O}^{\oplus(n-1)} \arrow[r, tail] & \mathcal{O}^{\oplus(n+1)} \arrow[r, two heads] & \mathcal{O}^{\oplus 2} \end{tikzcd}$$

where $\mathcal{F}^\vee = \mathcal{T}$. The bottom injective arrow corresponds to the $(n-1)$-dimensional subspace of equations cutting the line ℓ_0. The top horizontal sequence is exact and defines $\mathcal{U}_0$ over $\mathbb{G}\mathrm{r}[v,n] \setminus \ell_0^\star$. In particular, we have

$$c_1\mathcal{U}_0 = c_1\mathcal{F}. \tag{27}$$

Let us introduce next the divisor class in $\mathbb{X}'''$ corresponding to the condition that a *csc* be incident to a line. It is the class of the strict transform of a divisor $\mathbb{D} \subset \mathbb{X}$ defined as the closure of the set of $\mathrm{x} = (x, \ell, \langle q_1, q_2\rangle)$ in $\mathbb{X}$ such that

(1) the line $\ell_0'(x)$ (25) is defined,
(2) gcd $(q_1, q_2) = 1$, and
(3) the associated *csc* residual to $\ell^\perp$ meets $\ell_0'(x)$.

Consider first the divisor,

$$\widetilde{\mathbb{D}} = \{(x, \ell, \langle q_1, q_2\rangle) | \; q_1 \cap q_2 \cap \ell_0'(x) \neq \emptyset\} \subset \mathbb{X}.$$

Note that, if f_1, f_2 denote homogeneous polynomials of degree m and ℓ_0 is a line in $\mathbb{P}^4$, then $f_1 \cap f_2 \cap \ell_0 \neq \emptyset$ if and only if there are homogeneous polynomials g_1, g_2 of degree $m-1$ such that

- the polynomial $g_1 f_1 + g_2 f_2$ of degree $2m-1$ vanishes along ℓ_0, and
- either $g_1 \neq 0$ or $g_2 \neq 0$ on ℓ_0.

If $\mathcal{Q}_0$ denotes the space of linear forms on the line ℓ_0, the preceeding discussion shows that $f_1 \cap f_2 \cap \ell_0 \neq \emptyset$ holds if and only if the map of vector spaces of rank $2m$,

$$\langle f_1, f_2\rangle \otimes S_{m-1}\mathcal{Q}_0 \longrightarrow S_{2m-1}\mathcal{Q}_0$$

induced by multiplication is of rank $< 2m$. Thus, if the exact sequence $\mathcal{U}_0 \rightarrowtail \mathcal{F} \twoheadrightarrow \mathcal{Q}_0$ defines the section (25) off some codimension-2 subset, we see that the divisor $\widetilde{\mathbb{D}}$ is given by the condition that the map of rank-4 vector bundles

$$\mathcal{A} \otimes \mathcal{Q}_0 \longrightarrow S_{4-1}\mathcal{Q}_0$$

(see (9)) be of rank < 4. This yields the formula for the class,

$$[\widetilde{\mathbb{D}}] = c_1(\overset{4}{\wedge}(\mathcal{A} \otimes \mathcal{Q}_0)^\vee \otimes \overset{4}{\wedge} S_3\mathcal{Q}_0) \cap \mathbb{X} = (4c_1(\mathcal{Q}_0) - 2c_1(\mathcal{A})) \cap \mathbb{X}.$$

One checks that $\widetilde{\mathbb{D}}$ contains (simply) the pullback of the divisor $\mathbb{D}_{\ell_0} \subset \mathbb{G}\mathrm{r}(2, \mathcal{F})$ defined by the condition of incidence of a codimension 2 subspace $\ell^\perp$ to ℓ_0 fiberwise. Using (26), we find

$$[\mathbb{D}_{\ell_0}] = [\Sigma^1_{\ell_0}] = c_1\mathcal{Q} + c_1\mathcal{Q}_0 - c_1\mathcal{F}.$$

Thus we have

$$[\mathbb{D}] = [\widetilde{\mathbb{D}}] - [\mathbb{D}_{\ell_0}] = (3c_1(\mathcal{Q}_0) - 2c_1(\mathcal{A}) - c_1\mathcal{Q} + c_1\mathcal{F}) \cap \mathbb{X}.$$

For our specific example, we have $\mathcal{Q}_0 = \mathcal{O}^{\oplus 2}$, hence the above expression for $[\mathbb{D}]$ reduces to

$$[\mathbb{D}] = (c_1\mathcal{F} - 2c_1(\mathcal{A}) - c_1\mathcal{Q}) \cap \mathbb{X}.$$

It turns out that the total transform of $\mathbb{D}$ in $\mathbb{X}'$ contains the exceptional divisor $\mathbb{E}'$ once. More precisely, local computations along suitable coordinate charts show that $\mathbb{D} = \mathbb{D}' + \mathbb{E}'$. Similarly, we find $\mathbb{D}' = \mathbb{D}'' + \mathbb{E}''$ on $\mathbb{X}''$ and $\mathbb{D}'' = \mathbb{D}''' + \mathbb{E}'''$ on $\mathbb{X}'''$. Thus, we have to compute the dth self-intersection class of

$$[\mathbb{D}'''] = [\mathbb{D}] - \mathbb{E}' - \mathbb{E}'' - \mathbb{E}''' \tag{28}$$

restricted over the codimension-2 cycle (24), with $d = 18 + \dim \mathbb{G}\mathrm{r}[v, n] = 5n - 2$.

5. Fixed points

As explained above, we wish to compute the degree

$$\int_{\mathbb{X}'''} ([\mathbb{D}] - \mathbb{E}' - \mathbb{E}'' - \mathbb{E}''')^d c_2(\mathcal{U}^\vee).$$

This can be done by means of Bott's residue (or localization) formula in equivariant cohomology, cf. [2]. Let us state the simplified version we need. Given a variety $\mathbb{V}$ endowed with a $\mathbb{C}^\star$-action, suppose the number of fixed points is finite. Let p be a polynomial in the Chern classes of equivariant vector bundles over $\mathbb{V}$ of weighted degree equal to the dimension of $\mathbb{V}$. Then we have

$$\textbf{Bott's formula:} \quad \int_{\mathbb{V}} p = \sum_{\mathrm{x}} \frac{p_{\mathrm{x}}^{eq}}{c_{\mathrm{x}}^{eq}(T\mathbb{V})}. \tag{29}$$

The sum ranges over the set of fixed points; the numerator p_{x}^{eq} denotes the equivariant class at the fixed point x, and likewise for the equivariant top Chern class of the tangent bundle, $T\mathbb{V}$, occurring in the denominator. These equivariant classes translate merely into appropriate symmetric functions on the weights of the fibers of the relevant bundles at each fixed point as we explain below. Each summand turns out to be a rational number; quite miraculously they add up to yield the integer in the left hand side of (29).

5.1 $\mathbb{C}^\star$-ACTION. Our immediate task is to describe, for the case at hand, the points fixed under a suitable $\mathbb{C}^\star$-action. We will eventually restrict to the model $\mathbb{X}'''\to X = \mathbb{G}\mathrm{r}[v,n]$, but for the time being we may as well fix a point x in the base variety X and work out our way up along the tower of blowups (10).

We let $\mathcal{F}$ now stand for the space of linear forms in the homogeneous coordinates $z_0,\ldots,z_4$ of $\mathbb{P}^4$. We assume the $\mathbb{C}^\star$-action is given by sufficiently general weights $w_0,\ldots,w_4$. This means that the action can be described by

$$t\cdot z_i = t^{w_i} z_i, \quad 0\le i\le 4, t\in\mathbb{C}^\star.$$

In practice, it is harmless to commit the notational abuse of using the same symbol for the homogeneous coordinates and their weights. The assumption that the weights be general means that we require that $(w_0,\ldots,w_4)$ lie off the zeros of certain finitely many explicit polynomials as shown in the sequel.

The grassmannian $\mathbb{G}\mathrm{r}(2,\mathcal{F})$ has the ten fixed points given by $\langle z_i, z_j\rangle$, with $0\le i<j\le 4$. For the sake of concreteness, we pick

$$\ell_0 = \langle z_0, z_1\rangle$$

throughout the discussion below.

The fiber of $\mathbb{X}$ over each of the ten fixed points in $\mathbb{G}\mathrm{r}(2,\mathcal{F})$ is the grassmannian $\mathbb{G}\mathrm{r}(2,\mathcal{F}_2^{\langle z_i,z_j\rangle})$. Here the fixed points correspond to the choices of pairs of monomials in the space of quadrics $\langle z_i,z_j\rangle\cdot\mathcal{F}$. There are nine monomials, totaling $\binom{9}{2}$ pencils. These must now be sorted according to their position with respect to the succeeding blowup centers. Thus, the fixed points in the corresponding fiber of $\mathbb{X}$ over ℓ_0 are given by the choices of any two monomials among the list

$$(\mathcal{F}_2^\perp)_{\ell_0} = \langle z_0, z_1\rangle\cdot\mathcal{F} = z_0^2, z_1z_0, z_2z_0, z_3z_0, z_4z_0, z_1^2, z_2z_1, z_3z_1, z_4z_1. \tag{30}$$

Referring to (11) (see also Fig. 2), we see that, in order that a point

$$(\ell_0, \langle hz_i, hz_j\rangle)\in\mathbb{X}$$

lie in the first blowup center, $\mathbb{Y}$, the common factor h must be either z_0 or z_1. As for ℓ', it can be taken as any one of the ten fixed subspaces $\langle z_i,z_j\rangle$. We see also that the fiber of $\mathbb{Y}_1$ (14) over ℓ_0 contains the fixed points $\langle z_iz_0, z_iz_1\rangle, 0\le i\le 4$. Of course the intersection of $\mathbb{Y}$ and $\mathbb{Y}_1$ over ℓ_0 contains precisely the two fixed points $\langle z_iz_0, z_iz_1\rangle$ with $i=0,1$. The remaining 13 fixed points, namely,

$$\begin{gathered}\langle z_1^2, z_0^2\rangle, \langle z_2z_1, z_0^2\rangle, \langle z_3z_1, z_0^2\rangle, \langle z_4z_1, z_0^2\rangle, \langle z_1^2, z_2z_0\rangle, \langle z_1^2, z_3z_0\rangle,\\ \langle z_1^2, z_4z_0\rangle, \langle z_2z_1, z_3z_0\rangle, \langle z_3z_1, z_2z_0\rangle, \langle z_4z_1, z_2z_0\rangle, \langle z_4z_1, z_3z_0\rangle,\\ \langle z_4z_0, z_3z_1\rangle, \langle z_4z_0, z_2z_1\rangle\end{gathered} \tag{31}$$

lie off the images of the blowup centers. Therefore, they contribute immediately to Bott's formula. In fact, these points correspond each to a well-defined point of $\mathbb{X}'''$ (see (19)); we dub them **terminal** fixed points. For instance, the pencil $\langle z_2z_1, z_3z_0\rangle$ enlarges to the net $\langle z_2z_1, z_3z_0, z_2z_3\rangle$ together with the flag of spaces of cubics $\langle z_2z_1, z_3z_0\rangle \cdot \mathcal{F} \subset \langle z_2z_1, z_3z_0, z_2z_3\rangle \cdot \mathcal{F}$.

Now consider a point, say $\langle z_0z_1, z_0z_2\rangle \in \mathbb{Y} \setminus (\mathbb{Y}_1 \cup \mathbb{Y}_2)$. The exceptional fiber is the projectivization of the quotient space $(\langle z_1, z_2\rangle \cdot \mathcal{F})/(h \cdot \ell')$ of quadrics vanishing on $\ell' = \langle z_1, z_2\rangle$, modulo the subspace $z_0\langle z_1, z_2\rangle$. We find that the fixed points correspond to the monomials $z_1^2, z_2z_1, z_3z_1, z_2^2, z_3z_2, z_4z_1, z_4z_2$. These are all terminal.

Let us work out the fiber of $\mathbb{E}'$ say over the point

$$\langle z_0z_0, z_0z_1\rangle \in \mathbb{Y} \cap \mathbb{Y}_1 = \mathbb{Y}_2 \cap \mathbb{Y}_1.$$

Arguing as before, we find the monomials, $z_0z_2, z_0z_3, z_0z_4, z_1^2, z_2z_1, z_3z_1, z_4z_1$. The first three yield a pair of points in $\mathbb{Y}_1' \cap \mathbb{Y}_2'$, whereas the remaining four are terminal.

Look at the point (cf. (16))

$$\mathrm{y}' = (\mathrm{p}, \ell, h) \leftrightarrow (\langle z_0, z_1\rangle, \langle z_0z_0, z_0z_1\rangle, \langle z_0z_0, z_0z_1, z_0z_2\rangle) \in \mathbb{Y}_1'. \tag{32}$$

We have the flag $\mathrm{p} = \langle z_0z_0, z_0z_1\rangle, \ell = \ell_0, h = z_0$. Let us examine the fiber of $\mathbb{E}''$ over y'. Consider the space $\mathcal{F}^{\ell+\mathrm{p}^2}$ spanned by the cubic monomials that vanish along the codimension-2 subspace ℓ and furthermore have zero gradient at the codimension-3 subspace p. There are 19 of these. The exceptional fiber here is the projectivization of the quotient space $(\mathcal{F}^{\ell+\mathrm{p}^2})/(h \cdot \ell \cdot \mathcal{F})$. There are the following fixed points,

$$\langle z_1^3, z_1^2z_2, z_1^2z_3, z_1^2z_4, z_0z_2^2, z_0z_2z_4, z_0z_2z_3, z_1z_2^2, z_1z_2z_3, z_1z_2z_4\rangle.$$

Which of these lie in $\mathbb{Y}_2''$? Looking at the description in (18), it is clear that

$$z_0z_2^2, z_0z_2z_3, z_0z_2z_4$$

are the only ones. Take $z_0z_2^2$; the corresponding point

$$\mathrm{y}'' = (\mathrm{p}, \mathrm{o}, \ell, h, \ell') \tag{33}$$

can be read from (32) taking into account the expression for $h \cdot \mathcal{F}_2^{(\mathrm{o}+\mathrm{p},\ell')} = \langle z_0z_2^2\rangle + h \cdot \mathcal{F}_2^{\ell}$. Thus we find $\ell = \ell' = \langle z_0, z_1\rangle, \mathrm{o} = \mathrm{p} = \langle z_0, z_1, z_2\rangle$.

Let us work out now the fiber of the last exceptional divisor, $\mathbb{E}''' \to \mathbb{Y}''_2$, over the point y$''$ considered just above. In view of (22), we find that the fiber is the projectivization of the space spanned by the 8 monomials,

$$\langle z_1z_2z_3, z_1z_2z_4, z_1^2z_3, z_1^2z_4, z_2^3, z_1z_2^2, z_1^2z_2, z_1^3\rangle.$$

Using MAPLE, we find a total of 489 fixed points on $\mathbb{X}'''$ sitting over the distinguished line ℓ_0.

5.2 Weights. Our job now is to describe the weights of the relevant induced $\mathbb{C}^\star$-representation at each of the fixed points. This requires tracing up the weight decomposition of the fibers of some natural equivariant bundles that contribute in Bott's residue formula.

It is also convenient to write the weight decomposition in the form

$$\mathcal{F} = t^{z_0} + \ldots + t^{z_4}. \tag{34}$$

Each summand t^{z_i} above stands for the eigensubspace of $\mathcal{F}$ where the action is given by the indicated character. In the final step, we must replace each z_i by suitable functions of the original weights, say $x_0, \ldots, x_n$, of the $\mathbb{C}^\star$-action on $\mathbb{P}^n$, keeping track of the relevant induced actions.

The contribution of the codimension-2 cycle $[\Sigma^2_{x_0}]$ (24) at the fixed point $\ell_0 = \langle z_0, z_1\rangle \in \mathbb{G}\mathrm{r}(2, \mathcal{F})$, can be read from the weight decomposition of the corresponding fibers of the tautological bundles in (5), to wit,

$$\begin{aligned} \mathcal{U}_{\ell_0} &= t^{z_0} + t^{z_1} \\ \mathcal{Q}_{\ell_0} &= t^{z_2} + t^{z_3} + t^{z_4}. \end{aligned} \tag{35}$$

The equivariant Chern class for our pet example is

$$c_2^{eq}(\mathcal{U}) = z_0z_1. \tag{36}$$

We write next the weight decomposition for the induced action on the tangent bundle of $\mathbb{G}\mathrm{r}(2, \mathcal{F})$:

$$\begin{aligned} T\mathbb{G}\mathrm{r}(2, \mathcal{F})_{\ell_0} &= \mathrm{Hom}(\mathcal{U}_{\ell_0}, \mathcal{Q}_{\ell_0}) \\ &= \mathcal{U}^\vee_{\ell_0} \otimes \mathcal{Q}_{\ell_0} \\ &= (t^{-z_0} + t^{-z_1})(t^{z_2} + t^{z_3} + t^{z_4}) \\ &= t^{z_2-z_0} + t^{z_2-z_1} + t^{z_3-z_0} + t^{z_3-z_1} + t^{z_4-z_0} + t^{z_4-z_1}. \end{aligned} \tag{37}$$

Consider for instance the terminal point $\mathrm{x} = \langle z_2 z_1, z_3 z_0\rangle \in \mathbb{X} = \mathbb{G}\mathrm{r}(2, \mathcal{F}_2^{\perp})$ (see (6),)-(8)), picked in the fiber over ℓ_0. Recalling (30) and (9) we get

$$
\begin{aligned}
T\mathbb{X}_{\mathrm{x}} =&\mathrm{Hom}(\mathcal{A}, \mathcal{Q}_{\mathcal{A}})_{\mathrm{x}} + T\mathbb{G}\mathrm{r}(2, \mathcal{F})_{\ell_0} + TX_x \\
=&t^{(z_0-z_3)} + t^{(2z_0-z_2-z_1)} + t^{(z_1-z_3)} + t^{(z_0-z_2)} + t^{(z_2-z_3)} \\
&+ t^{(z_0-z_1)} + t^{(z_4-z_3)} + t^{(z_0+z_4-z_2-z_1)} + t^{(2z_1-z_3-z_0)} + t^{(z_1-z_2)} \\
&+ t^{(z_1-z_0)} + t^{(z_3-z_2)} + t^{(z_4+z_1-z_3-z_0)} + t^{(z_4-z_2)} + t^{(z_2-z_0)} \\
&+ t^{(z_2-z_1)} + t^{(z_3-z_0)} + t^{(z_3-z_1)} + t^{(z_4-z_0)} + t^{(z_4-z_1)} + TX_x.
\end{aligned}
\tag{38}
$$

Let ξ be the top equivariant Chern class of TX at the image x of x. The denominator of the term corresponding to the fixed point x in (29) is ξ times the product of the 20 exponents appearing in (38), namely:

$$
\begin{gathered}
(z_0-z_1)^2(z_0-z_2)^2(z_0-z_3)^2(z_1-z_2)^2(z_1-z_3)^2(z_2-z_3)^2 \\
(2z_0-z_2-z_1)(z_0+z_4-z_2-z_1)(2z_1-z_3-z_0)(z_4+z_1-z_3-z_0) \\
(z_4-z_0)(z_4-z_1)(z_4-z_2)(z_4-z_3)\cdot\xi.
\end{gathered}
\tag{39}
$$

The contribution ξ coming from the tangent space to the base $X = \mathbb{G}\mathrm{r}[v, n]$ is shown next. For simplicity, we stick to the case $n = 6, b = 2, v = 1$, i.e., cones in $\mathbb{P}^6$ over a *csc* with a vertex of dimension 1.

Say $x_0, \ldots, x_n$ are homogeneous coordinates in $\mathbb{P}^n$, and let the $\mathbb{C}^{\star}$-action be given by sufficiently general weights, abusively written also as $x_0, \ldots, x_n$. Thus we write the weight decomposition

$$(\mathbb{C}^{n+1})^{\star} = \sum_0^n t^{x_i}.$$

The fixed points in $\mathbb{G}\mathrm{r}[v, n]$ are the $\binom{n+1}{v+1}$ unit points for the Plücker imbedding. Let us pick the vertex V_0 that corresponds to the subspace defined by $x_0, \ldots, x_4$. Then the weight decomposition of $\mathcal{T}_{V_0}^{\vee} \subset (\mathbb{C}^{n+1})^{\vee}$ is simply

$$\mathcal{T}_{V_0}^{\vee} = t^{x_0} + \cdots + t^{x_4}.$$

For the tangent bundle, we have that $T\mathbb{G}\mathrm{r}[1, 6] = \mathrm{Hom}(\mathcal{V}, \mathcal{T})$. Its fiber at p_0 is given by

$$(t^{x_5} + t^{x_6}) \cdot (t^{-x_0} + \cdots + t^{-x_4}) = \sum_{\substack{5\le i\le 6\\ 0\le j\le 4}} t^{x_i - x_j}. \tag{40}$$

Thus, in the previous discussion, starting at (34), we must replace the weights of $\mathcal{F}_{\mathrm{p}_0} = \mathcal{T}_{\mathrm{p}_0}^{\vee}$ by

$$z_i = x_i, \quad i = 0, \ldots, 4. \tag{41}$$

Correspondingly, we get the value

$$\xi = \prod_{\substack{5\le i\le 6\\ 0\le j\le 4}} (x_i - x_j) \tag{42}$$

to substitute in (39).

The numerator in Bott's formula (29) is a suitable homogeneous polynomial of the same degree (= 20 + 10) as the denominator. It corresponds to the selfintersection $(\mathbb{D}''')^{28}$ of the incidence divisor (28), times the codimension-2 cycle $[\Sigma^2_{x_0}]$, cf. (36). For the specific terminal fixed point under examination, we find the numerator

$$x_1x_0(x_2 + x_3 + 2x_4 + 2x_5)^{28}.$$

Let us work out the contribution of a point x''' which lies in $\mathbb{E}'''$, e.g.,

$$\mathrm{x}''' = [[z_0, z_0 + z_1, [z_0, z_1, z_2], [z_0, z_1, z_2]], z_1^3].$$

It is in the fiber over y'' (33). The right hand side above codifies

(1) the hyperplane $h = z_0$ containing the plane $\ell = \langle z_0, z_1\rangle$ cf. (14);

(2) the line o=p=$\langle z_0, z_1, z_2\rangle$, so that (p,$\ell, h$) lies in $\mathbb{Y}'_1$ and (p,o,ℓ, h, ℓ) lies in $\mathbb{Y}''_2$; cf. (16), (18);

(3) the 13-dimensional space of cubics is $\langle z_1^3\rangle + h\cdot\mathcal{F}_2^{\mathrm{p}}$, cf. (22).

We have by general principles

$$\begin{aligned} T\mathbb{X}'''_{\mathrm{x}'''} &= T\mathbb{E}'''_{\mathrm{x}'''} + \mathcal{O}(\mathbb{E}''')_{\mathrm{x}'''};\\ T\mathbb{E}'''_{\mathrm{x}'''} &= T\mathbb{Y}''_{\mathrm{x}''} + ((\mathcal{N}_{\mathbb{Y}''/\mathbb{X}''} - \mathcal{O}(\mathbb{E}''')) \otimes \mathcal{O}(\mathbb{E}'''))_{\mathrm{x}'''}. \end{aligned}$$

For the explicit determination of the weight of the exceptional line bundle, presently

$$\mathcal{O}(\mathbb{E}''')_{\mathrm{x}'''} = \frac{z_1^3}{z_2^2 z_0},$$

we first recall that the normal bundle to the exceptional divisor

$$\mathbb{E}''' = \mathbb{P}\left(\mathcal{N}_{\mathbb{Y}''/\mathbb{X}''}\right)$$

is given by

$$\mathcal{O}(\mathbb{E}''')_{\mathbb{E}'''} = \mathcal{O}_{\mathbb{P}\left(\mathcal{N}_{\mathbb{Y}''/\mathbb{X}''}\right)}(-1).$$

Thus the fiber of $\mathcal{O}(\mathbb{E}''')_{\mathbb{E}'''}$ over x''' is the subspace of $(\mathcal{N}_{\mathbb{Y}''/\mathbb{X}''})_{\mathrm{y}''}$ that corresponds (tautologically) to x'''. In order to find it, we compare the following two descriptions of the exceptional fiber

$$\begin{aligned} \mathbb{E}'''_{\mathrm{x}'''} &= \mathbb{P}\left(z_1^3, z_2z_1^2, z_3z_1^2, z_4z_1^2, z_2^2z_1, z_3z_2z_1, z_4z_2z_1, z_2^3\right)\\ &= \mathbb{P}\left(\mathcal{N}_{\mathbb{Y}''/\mathbb{X}''}\right)_{\mathrm{y}''}. \end{aligned} \tag{43}$$

The above monomials come from, and yield a basis for (22); the correct attribution of weights stems from the knowledge of

$$(44)\quad (\mathcal{N}_{\mathbb{Y}''/\mathbb{X}''})_{\mathrm{y}''} = \frac{z_2}{z_0} + \frac{z_1}{z_0} + \frac{z_1^3}{z_2^2 z_0} + \frac{z_1^2}{z_2 z_0} + \frac{z_3 z_1^2}{z_2^2 z_0} + \frac{z_4 z_1^2}{z_2^2 z_0} + \frac{z_3 z_1}{z_2 z_0} + \frac{z_4 z_1}{z_2 z_0},$$

where each fraction represents an eigenspace, e.g., ,

$$(z_3 z_1^2)/(z_2^2 z_0) \leftrightarrow t^{z_3+2z_1-2z_2-z_0}.$$

In order to find out which monomial corresponds to one of the eight eigenspaces indicated in the above decomposition of the normal fiber, we compare the induced weights on the tangent space of $\mathbb{E}'''_{\mathrm{x}'''}$ at each of them, using the two recipes in (43). For instance, we have, on the one hand,

$$(45)\quad T_{(z_1^3)}\mathbb{P}\left(z_1^3, z_2 z_1^2, z_3 z_1^2, z_4 z_1^2, z_2^2 z_1, z_3 z_2 z_1, z_4 z_2 z_1, z_2^3\right) = \\ (z_2 z_1^2 + z_3 z_1^2 + z_4 z_1^2 + z_2^2 z_1 + z_3 z_2 z_1 + z_4 z_2 z_1 + z_2^3)/z_1^3.$$

Similarly, for each summand

$$\xi \in \{\frac{z_2}{z_0}, \frac{z_1}{z_0}, \cdots, \frac{z_4 z_1}{z_2 z_0}\}$$

occurring in (44), we may write

$$T_\xi \mathbb{P}\left(\mathcal{N}_{\mathbb{Y}''/\mathbb{X}''}\right)_{\mathrm{y}''} = ((\mathcal{N}_{\mathbb{Y}''/\mathbb{X}''})_{\mathrm{y}''} - \xi)/\xi.$$

Precisely for $\xi = z_1^3/(z_2^2 z_0)$ the weights occurring in the right hand side of the last display matches the expression (45). This kills the indeterminacy of a possible twist by a line bundle arising from the identification (43). It yields the correct match $z_1^3 \leftrightarrow z_1^3/(z_2^2 z_0)$.

The full calculation is better left for a Maple script, cf. [9]. We find the following table for the first few values of vertex dimension. (We put for short *twc*=twisted cubic, *csc*=cubic surface, *seg*=Segre variety.)

base	vertex dimension	$\int(\mathbb{D}''')^d$
twc	0	279596220
twc	1	592593995880
twc	2	1009133690375160
csc	0	7129237833060
csc	1	56119331565638208
csc	2	393718789073347359072
seg	0	105250439124351864
seg	1	1850280711419134165248
seg	2	37561018375847420442743040

number of cones over cubic scrolls
incident to the maximal number of lines

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